UDC 531.381

ON PERIODIC SOLUTIONS OF THE EULER — POISSON EQUATIONS

PMM Vol. 43, No. 2, 1979, pp 262 - 266,

V. S. SERGEEV
(Moscow)

(Received June 13, 1978)

A novel family is shown of the periodic solutions of the equations of motion of a heavy rigid body about a fixed point, for the case when the mass distribution within the body is almost the same as the distribution in the case of the Lagrange integrability. The periodic solutions obtained for the unperturbed problem correspond to the case when one of the frequencies of the regular Lagrange precession becomes equal to zero, i.e. they correspond to the steady rotations about the axes situated in the principal plane of inertia of the body about the fixed point. These steady rotations correspond to a bifurcation, and the regular precessions branch out from them [1]. For such motions the variational equations of the problem stated above have two zero roots with a single group of solutions. The periodic solutions near to the steady rotations of an arbitrary solid, were studied in [2]. These solutions, as well as the solutions obtained in [3], were based on the Liapunov theorem, i.e. on the assumption that the frequencies of the corresponding system of equations in variations were incommensurable. The author of [4] used the Poincaré method to show the existence of periodic solutions generated by the steady rotations of the Euler case.

1. Let us consider the equations of motion of a heavy rigid body about a fixed point O, written in the Euler — Poisson variables

$$A dp / dt + (C - B) qr = mg (y_0\gamma'' - z_0\gamma') \tag{1.1}$$
$$d\gamma / dt = r\gamma' - q\gamma'' \quad (ABC, x_0y_0z_0, pqr, \gamma\gamma'\gamma'')$$

where we assume that the positive direction of the OZ -axis, stationary in the $OXYZ$ coordinate system, coincides with the direction of the force of gravity g.

We shall consider a class of motions of a rigid body similar to the regular precession of the Lagrange's case. With this in mind, we transform the equations (1.1) by introducing a small parameter μ using the following substitutions:

$$\begin{gathered} x_0 = \mu x_1 l, \quad y_0 = \mu y_1 l, \quad z_0 = z_1 l \\ B = A(1 + \mu D), \quad k^2 = mgl / C \\ p = kp_1, \quad q = kq_1, \quad r = k(r_0 + \mu r_1) \\ \gamma = \gamma_1, \quad \gamma' = \gamma_1', \quad \gamma'' = \gamma_0'' + \mu\gamma_1'', \quad t = t_1 k^{-1} \end{gathered} \tag{1.2}$$

where l denotes the characteristic dimension of the body, e.g. $l^2 = C / m$, while D, $r_0 \neq 0$ and $\gamma_0'' \neq 0$ are constants.

Having obtained the transformed equations, we perform a further change of variables, namely $\gamma_1, \gamma_1', p_1, q_1 \to \gamma_2, \gamma_2', p_2, q_2$

$$\gamma_1 = -\gamma_0''M \cos \Omega t_1 + \mu\gamma_2, \quad p_1 = (\Omega - r_0) M \cos \Omega t_1 + \mu p_2 \qquad (1.3)$$
$$\gamma_1' = \gamma_0''M \sin \Omega t_1 + \mu\gamma_2', \quad q_1 = (r_0 - \Omega) M \sin \Omega t_1 + \mu q_2$$

where M is an arbitrary constant and Ω denotes one of the frequencies of the regular precession of the Lagrange's case. This yields a nonautonomous system of differential equations in which the right-hand sides are holomorphic with respect to the new variables and μ, and are periodic in time. When the conditions $z_0 \neq 0, \Omega \neq 0, r_0 \neq 0, \gamma_0'' \neq 0, \pm 1$ all hold, then the above system of equations admists a family of periodic solutions with the period $T = 2\pi / (k\Omega)$ (*).

Let us consider a degenerate case of regular precession, i. e. assume that $\Omega = 0$ in (1.3), or in other words, we require that the equation

$$Cz_1\gamma_0'' + (C - A) r_0^2 = 0 \qquad (1.4)$$

holds when $z_1 \neq 0$ and $C - A \neq 0$.
Performing the substitutions (1.2) — (1.4), we obtain the following system of equations:

$$\begin{aligned} &dp_2 / dt_1 - A_1 r_0 q_2 + A_2 z_1 \gamma_2' = g_1 \\ &dq_2 / dt_1 + A_1 r_0 p_2 - A_2 z_1 \gamma_2 = g_2 \\ &d\gamma_2 / dt_1 - r_0 \gamma_2' + \gamma_0'' q_2 = g_3 \\ &d\gamma_2' / dt_1 - \gamma_2'' p_2 + r_0 (M^2 + 1) \gamma_2 = g_4 \\ &A_1 = (A - C) / A, \quad A_2 = C / A \end{aligned} \qquad (1.5)$$

and complement this system with the equation

$$dr_1 / dt_1 - y_1 \gamma_0'' M = g_5 \qquad (1.6)$$

and relation

$$\gamma_1'' = M\gamma_2 - \frac{\mu}{2\gamma_0''} (\gamma_2^2 + \gamma_2'^2 + \gamma_1''^2) \qquad (1.7)$$

obtained form the geometrical integral of (1.1) by recalling that the choice of the constant M is based on the relation $\gamma_0''^2 (M^2 + 1) = 1$.

In (1.5) and (1.6) g_i $(i = 1,. . .,5)$ are known holomorphic functions of the parameter μ and variables $\gamma_2, \gamma_2', p_2, q_2$ and r_1, becoming constants when $\mu = 0$.

2. Using the Poincaré's method of small parameter, we shall seek periodic solutions of (1.5), (1.6) in the form of power series in μ with the period almost equal to the period of the generating periodic solution of the system (1.5). With this in mind, we set

$$t_1 = (1 + \alpha) \tau, \quad \alpha = \mu\alpha' = \mu\alpha_1 + \mu^2\alpha_2 + \ldots \qquad (2.1)$$

In the course of computing every approximation in μ, the equation (1.6) becomes separated from the basic system (1.5). We shall assume that $r_1 = 0$ when $\tau = 0$. Applying the transformation

---

*) See Sergeev, V. S. On a family of periodic solutions of the equations of motion of a solid with a fixed point. The Third All-Union Chetaev Conference on the Stability of Motion, Analytical Mechanics and Motion Control. Annot. dokl. Irkutsk, 1977.

$$X_k = a_k p_2 + b_k \gamma_2, \quad X_{k+1} = a_{k+1} q_2 + b_{k+1} \gamma_2', \qquad k = 1, 3$$
$$a_1 = \varkappa_1 A_1 (A_1 + 1) r_0^2, \quad b_1 = A_1 r_0^3 \gamma_0''^{-3} (1 + A_1 \gamma_0''^2)$$
$$a_2 = \varkappa_1^2 A_1 r_0, \quad b_2 = -\varkappa_1^2 A_1 r_0^2 \gamma_0''^{-1}$$
$$a_3 = r_0, \quad b_3 = A_1 r_0^2 \gamma_0''$$
$$a_4 = -r_0^2 \gamma_0''^{-2} (1 + A_1 \gamma_0''^2), \quad b_4 = -r_0^3 \gamma_0''^{-1} A_1 (A_1 + 1)$$

we reduce (1.5) to the form

$$dX_1 / d\tau = \varkappa_1 X_2 + F_1, \quad dX_3 / d\tau = F_3 \tag{2.2}$$
$$dX_2 / d\tau = -\varkappa_1 X_1 + F_2, \qquad dX_4 / d\tau = \varkappa_2 X_3 + F_4$$

Here the functions F_i of the variables X_j $(i, j = 1, \ldots, 4)$ and μ are of the same type as g_i in (1.5). The constants

$$\varkappa_1^2 = 4A_2 z_1 \gamma_0'' + r_0^2 (M^2 + A_2^2), \quad \varkappa_2 = A_1 r_0^2 M^2 \tag{2.3}$$

are assumed to be different from zero, i.e. we also require that $\gamma_0'' \neq \pm 1$.

3. We write the $T' = 2\pi / \varkappa_1$ -periodic solutions of the autonomous system of equations (2.2) as follows:

$$\begin{aligned} X_1(\tau) &= (M_1 + \beta_1) \cos \varkappa_1 \tau + X_1'(\tau, \mu) \\ X_2(\tau) &= -(M_1 + \beta_1) \sin \varkappa_1 \tau + X_2'(\tau, \mu) \\ X_3(\tau) &= M_3 + \beta_3 + X_3'(\tau, \mu) \\ X_4(\tau) &= \varkappa_2 (M_3 + m_3 + \beta_3) \tau + M_4 + \beta_4 + X_4'(\tau, \mu) \end{aligned} \tag{3.1}$$

The constants M_k as well as the functions $\beta_k = \beta_k(\mu)$ $(k = 1, 3, 4)$ must be found, together with $\alpha = \alpha(\mu)$, from the conditions of periodicity. The functions $X_i'(\tau, \mu)$ appearing in (3.1) are found, one after the other, in the form of power series in μ, in the course of solving the system (2.2), (1.6). In addition we have $X_i'(\tau, 0) = m_i$ and

$$m_1 = -A_1 A_2 r_0 \gamma_0'' \varkappa_1 x_1, \quad m_2 = m_4 = 0$$
$$m_3 = -A_2 (1 + A_1 \gamma_0''^2)(A_1 \gamma_0'' M^2)^{-1} x_1$$

The equations (2.2), (1.6) admit two independent integrals corresponding to the energy and area integrals of the problem. These integrals, which become dependent when $\mu = 0$, nevertheless enable us to eliminate from the conditions of periodicity

$$X_i(T') - X_i(0) = 0, \quad i = 1, \ldots, 4 \tag{3.2}$$
$$r_1(T') - r_1(0) = 0 \tag{3.3}$$

the conditions of periodicity of the functions X_1 and X_3 provided that the inequality

$$x_1 [(2A - C)(1 + A_1 \gamma_0''^2) r_0^2 + C x_1 \gamma_0''^2] \neq 0 \tag{3.4}$$

holds.

The last condition of (3.2) serves to determine M_3 ($M_3 = -m_3$) and the holomorphic function $\beta_3 = B_3(\mu, \alpha', \beta_1, \beta_4)$, and we have $B_3(0, \alpha', \beta_1, \beta_4) = 0$. The condition (3.3) which can hold only when $y_1 = 0$, is reduced to the form

$$-T'd\,[\varkappa_2 (M_3 + m_3 + \beta_3)\, T' / 2 + M_4 + \beta_4] + \mu H = 0 \qquad (3.5)$$

$$d = A_2^{-1} D r_0 M\,(1 + A_1 \gamma_0''^2)\,(A_1 \gamma_0'' \varkappa_1^4)^{-1} + x_1 \varkappa_1^{-2}$$

where H is a holomorphic function of μ in the neighborhood of $\mu = 0$. When

$$d \neq 0 \qquad (3.6)$$

equation (3.5) yields $M_4 = 0$ and the holomorphic function $\beta_4 = B_4(\mu, \alpha', \beta_1)$ such that $B_4(0, \alpha', \beta_1) = 0$. Finally, the condition that X_2 is periodic when $M_1 \neq 0$ yields α' in the form of a power series in μ, the first term of which is equal to

$$\alpha_1 = \frac{r_0^4}{2\varkappa_1^4}\left[De^2 + \frac{x_1 M A_2}{\varkappa_1^2 \gamma_0''}(3\varkappa_1^2 r_0^{-2} + \gamma_0''^2 e^2)\right] \qquad (3.7)$$

$$(e = A_1 (1 + A_1))$$

Thus we can satisfy all conditions of periodicity (3.2), (3.3) and determine the functions $\alpha = \alpha(\mu)$, $\beta_s = \beta_s(\mu)$ $(s = 3, 4)$ in the form of series in positive integral powers of μ. The quantity $M_1 + \beta_1$ remains arbitrary. Let us require that $\gamma'' = \gamma_0''$ at the initial instant of time. Then M_1 and $\beta_1 = \beta_1(\mu)$ can be found from the relations (1.7) taken at $\tau = 0$. In this case we have

$$M_1 = A_2 \gamma_0''^3 \varkappa_1^3 x_1 (\gamma_0''^2 - 1)^{-1}$$

and this proves the following theorem.

T h e o r e m. When the mass distribution in the body is resembles that of the case of Lagrange integrability, i.e. when $A - B \sim \mu$, $A \neq C$, $x_0 \neq 0 \sim \mu$, $y_0 = 0$, $z_0 \neq 0$, where μ is small, then the equations of motion (1.1) admit a family of periodic solutions provided that $r_0 \neq 0$, $\gamma''_0 \neq 0, \pm 1$, and the conditions (1.4), (3.4) and (3.6) hold. The solutions can be written in terms of the holomorphic functions of the parameter μ in the neighborhood of $\mu = 0$, $T = 2\pi(1 + \alpha)/(k\varkappa_1)$ -periodic in t, with $\varkappa_1$ and α given by the formulas (2.1), (2.3) and (3.7).

N o t e, 1. Periodic solutions analogous to those obtained exist also in the case when the requirement of the theorem that $y_0 = 0$ is replaced by the condition that $y_0 \sim \mu^2$.

N o t e 2. If we replace, in the transformation (1.3), t_1 by $t_1 + \pi/(2\Omega)$, then we can show, as before, that a family of periodic solutions exists when the center of gravity of the body in question is situated so that $y_0 \sim \mu$ and $x_0 = 0$ (or, according to Note 1, $\sim \mu^2$).

4. Let us now explain to which motions the periodic solutions of (1.1) obtained above, correspond. To do this, we compute the first terms of expansions into power series in μ of the angle of nutation ϑ, the rates of characteristic rotation and precession $\varphi^\cdot$ and $\psi^\cdot$. We have

$$\vartheta = \vartheta_0 + \mu\Theta\,(1 + \cos \varkappa_1 \tau) + \ldots \quad (\gamma_0'' = \cos \vartheta_0) \qquad (4.1)$$

$$d\varphi / d\tau = \mu\Phi \cos \varkappa_1\tau + \ldots$$

$$d\psi / d\tau = (1+\alpha)\frac{r_0}{\gamma_0''} + \frac{\mu}{\gamma_0''}[(R - \Theta \operatorname{tg} \vartheta_0)(\cos \varkappa_1\tau - 1) - \Phi \cos \varkappa_1\tau] + \ldots$$

$$\Theta = \frac{A_2\gamma_0''^2 x_1}{A_1 M r_0^2 \sin \vartheta_0}, \quad \Phi = \frac{M_1(1 + A_1\gamma_0''^2)}{M A_1 \gamma_0''^2 \varkappa_1^5}$$

$$R = M_1 (\varkappa_1^5 A_1 \gamma_0'')^{-1} [D A_2^{-1} e r_0 (r_0 - \Omega) \gamma_0'' - x_1 (1 + A_1\gamma_0''^2)]$$

The first and third formula of (4.1) imply that the Oz -axis of the coordinate system associated with the principal axes of inertia of the body describes, in the first approximation, a small ellipse on the unit sphere with center at O rotating about the vertical OZ with constant angular velocity $n = k\,[r_0 - \mu\,(R - \Theta \operatorname{tg} \vartheta_0)] / \gamma_0''$. At the same time the body executes, in accordance with the second formula of (4.1), small librations about the Oz -axis. However, the angle φ acquires a secular term already in the second approximation. The family of motions depends on three independent constants, namely on the initial values ϑ_0, φ_0 and ψ_0 of the Euler angles, since the initial value of the variable

$$r = k\,[r_0 + \mu R\,(\cos \varkappa_1\tau - 1) + \ldots]$$

is connected with ϑ_0 by the formula (1.4).

The series representing the periodic solutions converge for sufficiently small values of μ. A method described in [5] can be used to obtain a guaranteed estimate of the radius of convergence of these series.

REFERENCES

1. Rubanovskii, V. N. On the bifurcation and stability of the steady rotations of a heavy solid with a fixed point. Teoretichna i prilozhna mekhanika, Sofiia, No. 2, 1974.

2. Mettler, E. Periodische und asymptotische Bewegungen des unsymmetrischen schweren Kreisels. Math. Z., Bd 43, 1937.

3. Iakh'ia, Kh. M. On periodic, almost stationary motions of a rigid body about a fixed point. PMM Vol. 41, No. 3, 1977.

4. Kozlov, V. V. New periodic solutions for the problem of motion of a heavy solid body around a fixed point. PMM Vol. 39, No. 3, 1975.

5. Sergeev, V. S. A case of periodic solutions of the equaions of motion of a heavy rigid body about a fixed point. Vestn. MGU, Ser. matem., mekhan., No. 6, Moscow, 1969.

Translated by L. K.